

GEOMETRY OF THE CLOSED UNIT BALL OF THE SPACE OF BILINEAR FORMS ON ℓ_∞^2

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ABSTRACT. We obtain all extreme and exposed points of the closed unit ball of the space of bilinear forms $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$. We also show that any (norm one) bilinear form $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$ for which the optimal constant of the Littlewood's 4/3 inequality is achieved is necessarily an extreme point. In the case of complex scalars we combine analytical and numerical evidences supporting that, at least for complex bilinear forms with real coefficients, the optimal constant of Littlewood's 4/3 inequality seems to be the trivial, i.e., 1.

1. INTRODUCTION

Given a Banach space E and a convex set $A \subset E$, a vector $x \in A$ is called an extreme point of A if $y, z \in A$ with $x = \frac{y+z}{2}$ implies $y = z$. The characterization of extreme points of certain Banach spaces is a fruitful subject of investigation (see [6, 7, 8, 11, 12] and references therein) and the identification of extreme points of the closed unit ball of certain spaces of polynomials and bilinear forms has been quite useful in certain optimization problems (see, for instance, [5]). For a detailed exposition of the subject we refer to [19]. In the present paper we present all extreme and exposed points of the closed unit ball of $\mathcal{L}({}^2\ell_\infty^2(\mathbb{R}))$, i.e., the space of all bilinear forms $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$. We also investigate the optimization problem in the closed unit ball $B_{\mathcal{L}({}^2\ell_\infty^2(\mathbb{R}))}$ of $\mathcal{L}({}^2\ell_\infty^2(\mathbb{R}))$ associated to the best constant of Littlewood's 4/3 inequality. More precisely we obtain all bilinear forms in the closed unit ball of $\mathcal{L}({}^2\ell_\infty^2(\mathbb{R}))$ such that the optimal constant of Littlewood's 4/3 inequality is achieved.

The paper is organized as follows. In Section 2 we obtain expressions for the norms of bilinear forms $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$ and $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{C}$ with real coefficients. In Section 3 these results are used to classify the extreme and exposed points of the closed unit ball of $\mathcal{L}({}^2\ell_\infty^2(\mathbb{R}))$. In Section 4 we use the results of Section 2 to present all bilinear forms $T : \ell_\infty^2 \times \ell_\infty^2 \rightarrow \mathbb{R}$ for which the optimal constant of the Littlewood's 4/3 inequality is achieved. More precisely, we obtain all bilinear forms satisfying the following optimization problem:

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$$\inf \left\{ \left(\sum_{j_1, j_2=1}^2 |T(e_{j_1}, e_{j_2})|^{\frac{4}{3}} \right)^{\frac{3}{4}}, \text{ among all 2-linear forms } T \in B_{\mathcal{L}(\ell_\infty^2(\mathbb{R}))} \right\} = \sqrt{2}.$$

In Section 5 we consider the complex version of this problem and, combining analytical and numerical approaches, we obtain strong evidence supporting that, at least for complex bilinear forms with real coefficients, the optimal constant of the Littlewood's 4/3 inequality seems to be the trivial, i.e., 1.

2. EXPRESSIONS FOR THE NORMS OF BILINEAR FORMS ON $\mathcal{L}(\ell_\infty^2(\mathbb{K}))$

A first step to determine the geometry of the unit ball of $\mathcal{L}(\ell_\infty^2(\mathbb{K}))$ is to find expressions for the norms. This is not a very pleasant task, mainly in the case of complex scalars.

Proposition 2.1. *Let $T : c_0 \times c_0 \rightarrow \mathbb{C}$ be given by $T(z, w) = \sum_{i,j=1}^2 a_{ij} z_i w_j$ with $a_{ij} \in \mathbb{R}$. Then*

(A)

$$\|T\| = \max \left\{ \begin{array}{l} |a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|, \\ \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21}(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}})}}}, \\ \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21}(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}})}}} \end{array} \right\}$$

if $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$ and $\text{sgn}\left(\frac{a_{11}a_{21}}{a_{12}a_{22}}\right) = -1$ and

$$\left| \frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2) \right| \leq \left| 2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}} \right) \right|;$$

(B)

$$\|T\| = \max \{ |a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}| \}$$

otherwise.

Proof. Note that

$$\|T\| = \sup \{ \|T_z\| : \|z\|_\infty = 1 \},$$

where $T_z : \ell_\infty^2(\mathbb{C}) \rightarrow \mathbb{C}$ is given by

$$T_z(w) = (a_{11}z_1 + a_{21}z_2)w_1 + (a_{12}z_1 + a_{22}z_2)w_2.$$

We thus have

$$\|T\| = \sup \{ \|T_z\| : \|z\|_\infty = 1 \} = \sup \{ |a_{11}z_1 + a_{21}z_2| + |a_{12}z_1 + a_{22}z_2| : \|z\|_\infty = 1 \}.$$

Hence, calculating $\|T\|$ is the same of maximizing the function

$$f(z) = |a_{11}z_1 + a_{21}z_2| + |a_{12}z_1 + a_{22}z_2|$$

with the restriction $\|z\|_\infty = 1$. Denoting $z_j = x_j + iy_j$, $j = 1, 2$, we have

$$\begin{aligned} f(z) &= \sqrt{(a_{11}x_1 + a_{21}x_2)^2 + (a_{11}y_1 + a_{21}y_2)^2} \\ &\quad + \sqrt{(a_{12}x_1 + a_{22}x_2)^2 + (a_{12}y_1 + a_{22}y_2)^2} \end{aligned}$$

Since $\|z\|_\infty = 1$, we can write $z_j = \cos \theta_j + i \sin \theta_j$, $j = 1, 2$. Hence

$$f(\theta_1, \theta_2) = \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \cos(\theta_1 - \theta_2)} + \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos(\theta_1 - \theta_2)}.$$

By making $t = \theta_1 - \theta_2$ we have

$$f(t) = \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \cos t} + \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t}.$$

- Proof of (A):

We divide the proof of (A) in two cases:

◦ First case. Suppose that $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$ and $a_{11} \neq \pm a_{21}$ and $a_{12} \neq \pm a_{22}$.

In this case, since $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$ and $a_{11} \neq \pm a_{21}$ and $a_{12} \neq \pm a_{22}$, f' always exists and

$$f'(t) = \frac{-a_{11}a_{21} \sin t}{\sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \cos t}} + \frac{-a_{12}a_{22} \sin t}{\sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t}} = 0$$

if and only if $t = k\pi, k \in \mathbb{Z}$ or

$$(1) \quad \frac{-a_{11}a_{21}}{a_{12}a_{22}} = \frac{\sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \cos t}}{\sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t}}.$$

Since $\operatorname{sgn}\left(\frac{a_{11}a_{21}}{a_{12}a_{22}}\right) = -1$, we have

$$(2) \quad 2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right) \cos t = \frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)$$

and since

$$\left| \frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2) \right| \leq \left| 2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right) \right|,$$

there is t_0 such that

$$\cos t_0 = \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right)}.$$

Thus

$$(3) \quad \|T\| = \max f = \max \left\{ \begin{array}{l} |a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|, \\ \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21}(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}})}} \\ + \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21}(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}})}} \end{array} \right\}.$$

◦ Second case. Suppose that $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$ and $(a_{11} = \pm a_{21} \text{ or } a_{12} = \pm a_{22})$.

In this case there are real numbers t_0 such that $f'(t_0)$ does not exist. For these values of t_0 we can see that

$$f(t_0) = |a_{11} + a_{21}| + |a_{12} + a_{22}|$$

or

$$f(t_0) = |a_{11} - a_{21}| + |a_{12} - a_{22}|.$$

For the values of t such that $f'(t)$ exists, we proceed as in the first case; therefore we also obtain (3).

• Proof of (B).

We consider three cases:

◦ Case 1. Suppose $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$, $a_{11} \neq \pm a_{21}$ and $a_{12} \neq \pm a_{22}$ with

$$\operatorname{sgn} \left(\frac{a_{11}a_{21}}{a_{12}a_{22}} \right) = 1.$$

From (1) we can observe that $f'(t) = 0$ if and only if $t = k\pi, k \in \mathbb{Z}$ and thus

$$\|T\| = \max f = \max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\}.$$

◦ Case 2. Suppose $(a_{11}, a_{21}, a_{12}, a_{22}) \in (\mathbb{R} \setminus \{0\})^4$, $a_{11} \neq \pm a_{21}$, $a_{12} \neq \pm a_{22}$,

$$\operatorname{sgn} \left(\frac{a_{11}a_{21}}{a_{12}a_{22}} \right) = -1$$

and

$$\left| \frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2) \right| > \left| 2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}} \right) \right|.$$

In this case, from (2) we also know that $f'(t) = 0$ if and only if $t = k\pi, k \in \mathbb{Z}$; therefore

$$\|T\| = \max f = \max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\}.$$

◦ Case 3. We may have one of the following situations:

(1) $a_{11}a_{21} = 0$ and $a_{12}a_{22} = 0$;

- (2) $a_{11}a_{21} = 0$ and $a_{12}a_{22} \neq 0$;
- (3) $a_{11}a_{21} \neq 0$ and $a_{12}a_{22} = 0$;
- (4) $a_{11}a_{21} \neq 0$ and $a_{12}a_{22} \neq 0$.

If we consider (1), f can be written as one of the following expressions:

- (a) $f(t) = |a_{11}| + |a_{12}|$;
- (b) $f(t) = |a_{11}| + |a_{22}|$;
- (c) $f(t) = |a_{21}| + |a_{12}|$;
- (d) $f(t) = |a_{21}| + |a_{22}|$.

We thus can write, in any case,

$$f(t) = |a_{11} + a_{21}| + |a_{12} + a_{22}| = |a_{11} - a_{21}| + |a_{12} - a_{22}|$$

and, of course, we obtain the expression of (B).

If we consider (2) there is no loss of generality in supposing $a_{11} = 0$. So, we get

$$f(t) = |a_{21}| + \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t}$$

and we consider two subcases.

◦ Subcase 1. If $a_{12} = a_{22}$ or $a_{12} = -a_{22}$, then there is a $t_0 \in \mathbb{R}$ such that $f'(t_0)$ does not exist. In this case, it is plain that

$$f(t_0) = |a_{21}| \leq |a_{11} + a_{21}| + |a_{12} + a_{22}|.$$

For other values of t we have

$$f'(t) = \frac{-a_{12}a_{22} \sin t}{\sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t}}$$

and thus $f'(t_1) = 0$ if and only if $t_1 = k\pi$ and $k \in \mathbb{Z}$. For these values of t_1 we have

$$f(t_1) = |a_{21}| + |a_{12} + a_{22}| = |a_{11} + a_{21}| + |a_{12} + a_{22}|$$

or

$$f(t_1) = |a_{21}| + |a_{12} - a_{22}| = |a_{11} - a_{21}| + |a_{12} - a_{22}|.$$

We thus have again the expression given in (B).

◦ Subcase 2. If $a_{12} \neq a_{22}$ and $a_{12} \neq -a_{22}$, then $a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \cos t \neq 0$ for all t and $f'(t)$ exists for all t ; thus we again obtain the expression of (B).

The situation (3) is similar to (2).

If we have (4) and ($a_{11} = \pm a_{21}$ or $a_{12} = \pm a_{22}$) we proceed as in the second case of (A). If $a_{11} \neq \pm a_{21}$ and $a_{12} \neq \pm a_{22}$ we are encompassed by Case 1 or Case 2 of (B). \square

For real scalars, for the obvious reasons, the expression of the norm is less complicated:

Proposition 2.2. *Let $T : c_0 \times c_0 \rightarrow \mathbb{R}$ be given by $T(x, y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$, with $a_{ij} \in \mathbb{R}$. Then*

$$\|T\| = \max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\}.$$

Proof. As in the proof of the complex case,

$$\|T\| = \sup\{\|T_x\| : \|x\|_\infty = 1\},$$

where $T_x : \ell_\infty^2(\mathbb{R}) \rightarrow \mathbb{R}$ is given by

$$T_x(y) = (a_{11}x_1 + a_{21}x_2)y_1 + (a_{12}x_1 + a_{22}x_2)y_2.$$

We thus have

$$\|T\| = \sup\{\|T_x\| : \|x\|_\infty = 1\} = \sup\{|a_{11}x_1 + a_{21}x_2| + |a_{12}x_1 + a_{22}x_2| : \|x\|_\infty = 1\}.$$

So, we shall maximize

$$f(x) = |a_{11}x_1 + a_{21}x_2| + |a_{12}x_1 + a_{22}x_2|$$

with the restriction $\|x\|_\infty = 1$. There is no loss of generality in supposing that $|x_1| = 1$. If $x_1 = 1$ (the case $x_1 = -1$ is similar), then

$$f(t) = |a_{11} + a_{21}t| + |a_{12} + a_{22}t|.$$

So, we shall maximize f under the restriction $|t| \leq 1$. Thus, invoking Lemma 6.2 the maximum of f is attained at $t = -1$ or when $t = 1$, and it is given by

$$\max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\}.$$

□

3. GEOMETRY OF THE UNIT BALL OF $\mathcal{L}(\ell_\infty^2(\mathbb{R}))$: EXTREME AND EXPOSED POINTS

As mentioned in the Introduction, given a Banach space E and a convex set $A \subset E$, a vector $x \in A$ is an *extreme point* of A if $y, z \in A$ with $x = \frac{y+z}{2}$ implies $y = z$. If $x \in A$ and there is a linear functional $f \in E^*$ such that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for all $y \in A \setminus \{x\}$, then x is called *exposed point*. It is not difficult to prove that exposed points are extreme points. In this section we obtain all extreme and exposed points of the closed unit ball of $\mathcal{L}(\ell_\infty^2(\mathbb{R}))$.

Theorem 3.1. *The extreme points of the closed unit ball of $\mathcal{L}(\ell_\infty^2(\mathbb{R}))$ are*

$$\begin{aligned} & \pm x_1y_1, \pm x_2y_1, \pm x_1y_2, \pm x_2y_2, \\ & \frac{1}{2}(\pm x_1y_1 \pm x_2y_1 \pm x_1y_2 \mp x_2y_2), \\ & \frac{1}{2}(\mp x_1y_1 \pm x_2y_1 \pm x_1y_2 \pm x_2y_2), \\ & \frac{1}{2}(\pm x_1y_1 \mp x_2y_1 \pm x_1y_2 \pm x_2y_2), \\ & \frac{1}{2}(\pm x_1y_1 \pm x_2y_1 \mp x_1y_2 \pm x_2y_2). \end{aligned}$$

Proof. For the sake of simplicity we shall denote $\ell_\infty^2(\mathbb{R})$ by ℓ_∞^2 along this proof. Let $T \in B_{\mathcal{L}(\ell_\infty^2)}$ be given by $T(x, y) = ax_1y_1 + bx_2y_1 + cx_1y_2 + dx_2y_2$. By symmetry, it suffices to consider the following cases, with $a, b, c, d \neq 0$:

- (1) $T(x, y) = ax_1y_1$;
- (2) $T(x, y) = ax_1y_1 + bx_2y_1$;
- (3) $T(x, y) = ax_1y_1 + bx_2y_1 + cx_1y_2$;
- (4) $T(x, y) = ax_1y_1 + bx_2y_1 + cx_1y_2 + dx_2y_2$.

Since $T \in B_{\mathcal{L}(\ell_\infty^2)}$, we know that $|a|, |b|, |c|$ and $|d|$ are not bigger than 1.

Case (1). If $|a| < 1$, let $0 < \varepsilon < 1 - |a|$. Defining

$$A(x, y) = (a + \varepsilon)x_1y_1$$

$$B(x, y) = (a - \varepsilon)x_1y_1,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus, T is not an extreme point. If $|a| = 1$, we can suppose $a = 1$. Thus, if there are $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$ such that $\frac{1}{2}(A + B) = T$, say,

$$A(x, y) = \alpha x_1y_1 + \beta x_2y_1 + \gamma x_1y_2 + \delta x_2y_2$$

$$B(x, y) = \alpha' x_1y_1 + \beta' x_2y_1 + \gamma' x_1y_2 + \delta' x_2y_2,$$

we have $(\alpha, \beta, \gamma, \delta) = (2 - \alpha', -\beta', -\gamma', -\delta')$. Since $|\alpha|, |\alpha'| \leq 1$, we conclude that $\alpha = \alpha' = 1$. Note that if $\beta \neq 0$, then $1 + \beta$ or $1 - \beta$ is bigger than 1. Estimating $A((1, 1), (1, 0))$ and $A((1, 1), (-1, 0))$ we conclude that $\|A\| > 1$ and the same happens to B ; therefore $\beta = 0$. The same argument shows us that $\gamma = \delta = 0$. Thus, T is an extreme point.

Case (2). Note that

$$\|T\| = |a| + |b| \leq 1.$$

Let $0 < \varepsilon < \min\{|a|, |b|\}$, and defining

$$A(x, y) = (a + \operatorname{sgn}(a)\varepsilon)x_1y_1 + (b - \operatorname{sgn}(b)\varepsilon)x_2y_1$$

$$B(x, y) = (a - \operatorname{sgn}(a)\varepsilon)x_1y_1 + (b + \operatorname{sgn}(b)\varepsilon)x_2y_1,$$

we conclude that $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus, T is not an extreme point.

Case (3). By Proposition 2.2, we have

$$\|T\| = \max\{|a + b| + |c|, |a - b| + |c|\}.$$

Note that

$$|a + b| + |c| = |a - b| + |c| \Leftrightarrow a = 0 \text{ or } b = 0.$$

Let us consider two subcases:

(3A) $ab > 0$;

(3B) $ab < 0$.

If (3A) happens, then $|a - b| + |c| < 1$. Defining $0 < \varepsilon < \frac{1-(|a-b|+|c|)}{2}$ and

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus T is not extreme point.

If (3B) happens, then $|a + b| + |c| < 1$. Defining $0 < \varepsilon < \frac{1-(|a+b|+|c|)}{2}$ and

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus T is not extreme point.

Case (4). We consider four subcases:

(4A) $ab > 0$ and $cd > 0$;

(4B) $ab < 0$ and $cd < 0$;

(4C) $ab > 0$ and $cd < 0$;

(4D) $ab < 0$ and $cd > 0$.

If (4A) happens, then $|a - b| + |c - d| < 1$. Considering $0 < \varepsilon < \frac{1-(|a-b|+|c-d|)}{2}$ and defining

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus T is not an extreme point.

If (4B) happens, then $|a + b| + |c + d| < 1$. Considering $0 < \varepsilon < \frac{1-(|a+b|+|c+d|)}{2}$ and defining

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus T is not an extreme point.

If (4C) happens we can assume $a, b, c > 0$ and $d < 0$. Note that by Proposition 2.2,

$$(4) \quad \|T\| = |a + b| + |c + d|$$

or

$$(5) \quad \|T\| = |a - b| + |c - d|.$$

We shall consider just (4) because (5) is similar. If we have (4) then, by Lemma 6.3, there are two possibilities:

$$(4CA) \quad a \geq -d, \quad b \geq -d \text{ and } c \geq -d;$$

$$(4CB) \quad a \geq c, \quad b \geq c \text{ and } -d \geq c.$$

We shall first prove that if

$$\text{card}\{a, b, c, -d\} \neq 1,$$

then T is not an extreme point. Let us first suppose (4CA).

If $\text{card}\{a, b, c, -d\} \neq 1$ we can assume $a \neq b$ because the other cases are analogous. We thus have two possibilities:

$$(4CAA) \quad a > b,$$

$$(4CAB) \quad a < b.$$

Let us first consider (4CAA):

Since $a > b$, we have $a > b \geq -d$ and $c \geq -d$. We consider two cases:

$$(4CAAA) \quad a > b > -d \text{ e } c \geq -d;$$

$$(4CAAB) \quad a > b = -d \text{ e } c \geq -d.$$

If (4CAAA) happens, since $a > b > -d > 0$ and $c \geq -d > 0$ we conclude that

$$a + b + c + d > a - b + c - d,$$

i.e.,

$$|a + b| + |c + d| > |a - b| + |c - d|$$

and thus, by Proposition 2.2, $|a - b| + |c - d| < 1$. Considering $0 < \varepsilon < \frac{1 - (|a - b| + |c - d|)}{2}$ e defining

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$. Thus T is not an extreme point.

If (4CAAB) happens, since $b = -d$ and using that $a > b > 0$ and $c \geq -d > 0$ we have

$$|a + b| + |c + d| = |a - b| + |c - d| = a + c$$

and, by then by Proposition 2.2,

$$\|T\| = a + c \leq 1.$$

We have two possibilities:

$$(4CAABA) \ a + c < 1;$$

$$(4CAABB) \ a + c = 1.$$

If (4CAABA) happens, we choose $0 < \varepsilon < \min\{a - b, 1 - (a + c)\}$ and define

$$A(x, y) = (a + \varepsilon)x_1y_1 + bx_2y_1 + cx_1y_2 - bx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + bx_2y_1 + cx_1y_2 - bx_2y_2,$$

and by Lemma 6.1, we conclude that $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$ and $\frac{1}{2}(A + B) = T$. Hence, T is not an extreme point.

If (4CAABB) happens, we can write

$$T(x, y) = ax_1y_1 + bx_2y_1 + (1 - a)x_1y_2 - bx_2y_2.$$

Since $c \geq -d$, it follows that

$$1 - a \geq b.$$

If $1 - a = b$, then

$$T(x, y) = (1 - b)x_1y_1 + bx_2y_1 + bx_1y_2 - bx_2y_2.$$

Note that

$$|(1 - b) - b| + |b - (-b)| = |1 - 2b| + 2b.$$

Since $1 - b = a > b > 0$, it follows that $0 < b < \frac{1}{2}$. Considering $0 < \varepsilon < \min\{b, \frac{a-b}{2}\}$ and defining

$$A(x, y) = (1 - b + \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + (b - \varepsilon)x_1y_2 + (-b + \varepsilon)x_2y_2$$

$$B(x, y) = (1 - b - \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + (b + \varepsilon)x_1y_2 + (-b - \varepsilon)x_2y_2,$$

we have $\frac{1}{2}(A + B) = T$ and $\|A\| = \|B\| = 1$. Hence, T is not an extreme point.

If $1 - a > b$, then $b < a < 1 - b$. Considering $0 < \varepsilon < \min\{a - b, 1 - a - b\}$ and defining

$$A(x, y) = (a + \varepsilon)x_1y_1 + bx_2y_1 + (1 - a - \varepsilon)x_1y_2 - bx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + bx_2y_1 + (1 - a + \varepsilon)x_1y_2 - bx_2y_2,$$

we conclude that $A, B \in B_{\mathcal{L}(\ell_\infty^2)}$ and $\frac{1}{2}(A + B) = T$. Thus T is not an extreme point.

Now let us prove (4CAB). Since $b > a$, then $b > a \geq -d$ and $c \geq -d$.

If $b > a > -d$ and $c \geq -d$, then

$$a > -d \Rightarrow a + d > -a - d \Rightarrow a + b + c + d > b - a + c - d.$$

Hence

$$|a + b| + |c + d| > |a - b| + |c - d|.$$

Considering $0 < \varepsilon < \frac{1-(|a-b|+|c-d|)}{2}$ and defining

$$A(x, y) = (a + \varepsilon)x_1y_1 + (b - \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2$$

$$B(x, y) = (a - \varepsilon)x_1y_1 + (b + \varepsilon)x_2y_1 + cx_1y_2 + dx_2y_2,$$

we conclude that $\frac{1}{2}(A + B) = T$ and $A, B \in B_{\mathcal{L}(^2\ell_\infty^2)}$. Thus T is not an extreme point.

If $b > a = -d$ and $c \geq -d$, then we shall proceed as in the case (4CAAB) to observe that

$$T(x, y) = ax_1y_1 + bx_2y_1 + cx_1y_2 - ax_2y_2$$

is not an extreme point.

So, it remains to look for extreme points in the case (4C) when

$$\text{card}\{a, b, c, -d\} = 1.$$

In this case we can write

$$(6) \quad T(x, y) = ax_1y_1 + ax_2y_1 + ax_1y_2 - ax_2y_2.$$

Since $2a = \|T\| \leq 1$, we have $a \leq \frac{1}{2}$. If $a < \frac{1}{2}$, T is not an extreme point. Let us show that when $a = \frac{1}{2}$ the bilinear form T given by (6) is an extreme point.

Suppose that there exist $A, B \in B_{\mathcal{L}(^2\ell_\infty^2)}$ such that $\frac{1}{2}(A + B) = T$. Denoting

$$A(x, y) = \alpha x_1y_1 + \beta x_2y_1 + \gamma x_1y_2 + \delta x_2y_2,$$

$$B(x, y) = \alpha' x_1y_1 + \beta' x_2y_1 + \gamma' x_1y_2 + \delta' x_2y_2,$$

we have

$$(\alpha + \alpha', \beta + \beta', \gamma + \gamma', \delta + \delta') = (1, 1, 1, -1).$$

Since $|\alpha|, |\alpha'| \leq 1$, it follows that $\alpha \in [0, 1]$. A similar argument tells us that $\beta, \gamma, -\delta \in [0, 1]$. We claim that if $\alpha \neq \frac{1}{2}$, then $A \notin B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))}$ or $B \notin B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))}$. Note that

$$\begin{aligned} 0 \leq \alpha < \frac{1}{2} &\Rightarrow -\frac{1}{2} < -\alpha \leq 0 \\ &\Rightarrow \frac{1}{2} < 1 - \alpha \leq 1 \\ &\Rightarrow \frac{1}{2} < \alpha' \leq 1 \end{aligned}$$

and

$$\frac{1}{2} < \alpha \leq 1 \Rightarrow 0 \leq \alpha' < \frac{1}{2}.$$

In a similar fashion,

$$0 \leq \beta < \frac{1}{2} \Rightarrow \frac{1}{2} < \beta' \leq 1$$

and so on.

So, let us first suppose $\alpha \in [0, \frac{1}{2})$. We may have $\beta \in [0, \frac{1}{2}]$ or $\beta \in (\frac{1}{2}, 1]$.

If $\beta \in [0, \frac{1}{2}]$, then $\alpha' \in (\frac{1}{2}, 1]$ and $\beta' \in [\frac{1}{2}, 1]$. Therefore,

$$B((1, 1), (1, 0)) = \alpha' + \beta' > 1$$

and thus $\|B\| > 1$, a contradiction.

If $\beta \in (\frac{1}{2}, 1]$, then we may have:

(P1) $\gamma \in [0, \frac{1}{2}]$ and $\delta \in [-1, -\frac{1}{2}]$;

(P2) $\gamma \in [0, \frac{1}{2}]$ and $\delta \in (-\frac{1}{2}, 0]$;

(P3) $\gamma \in (\frac{1}{2}, 1]$ and $\delta \in [-1, -\frac{1}{2}]$;

(P4) $\gamma \in (\frac{1}{2}, 1]$ and $\delta \in (-\frac{1}{2}, 0]$.

If (P1) holds, then $\alpha' \in (\frac{1}{2}, 1]$, $\gamma' \in [\frac{1}{2}, 1]$ and $\beta', -\delta' \in [0, \frac{1}{2}]$. Thus $\alpha' > \beta'$, $\gamma' \geq \beta'$ and $\alpha', \gamma' \geq -\delta'$.

When $\beta' \geq -\delta'$, by Lemma 6.3, we have

$$\|B\| = \alpha' + \beta' + \gamma' + \delta' > 1.$$

When $-\delta' \geq \beta'$, by Lemma 6.4, we have

$$\|B\| = \alpha' - \beta' + \gamma' - \delta' > 1.$$

If (P2) holds, then $\alpha', -\delta' \in (\frac{1}{2}, 1]$, $\gamma' \in [\frac{1}{2}, 1]$ and $\beta' \in [0, \frac{1}{2}]$. Thus $\alpha', -\delta' > \beta'$ and $\gamma' \geq \beta'$. By Lemma 6.4, we have

$$\|B\| = \alpha' - \beta' + \gamma' - \delta' > 1.$$

If (P3) holds, then

$$A((1, -1), (0, 1)) = -\delta + \gamma > 1$$

and thus $\|A\| > 1$.

If we have (P4), then $\beta, \gamma \geq \alpha$ and $\beta, \gamma \geq -\delta$. When $\alpha \geq -\delta$, by Lemma 6.3, we have

$$\|A\| = \alpha + \beta + \gamma + \delta > 1.$$

When $-\delta \geq \alpha$, by Lemma 6.4, we have

$$\|A\| = \beta - \alpha + \gamma - \delta > 1.$$

Now, let us suppose $\alpha \in (\frac{1}{2}, 1]$. We may have $\beta \in [0, \frac{1}{2}]$ or $\beta \in [\frac{1}{2}, 1]$. If $\beta \in [\frac{1}{2}, 1]$, then

$$A((1, 1), (1, 0)) = \alpha + \beta > 1$$

and hence $\|A\| > 1$, a contradiction. If $\beta \in [0, \frac{1}{2}]$, we may have:

$$(K1) \quad \gamma \in [0, \frac{1}{2}] \text{ and } \delta \in [-1, -\frac{1}{2}];$$

$$(K2) \quad \gamma \in [0, \frac{1}{2}] \text{ and } \delta \in (-\frac{1}{2}, 0];$$

$$(K3) \quad \gamma \in (\frac{1}{2}, 1] \text{ and } \delta \in [-1, -\frac{1}{2}];$$

$$(K4) \quad \gamma \in (\frac{1}{2}, 1] \text{ and } \delta \in (-\frac{1}{2}, 0].$$

If (K1) happens, then $\alpha > \beta$, $-\delta \geq \beta$ and $\alpha, -\delta \geq \gamma$. When $\beta \geq \gamma$, by Lemma 6.3, we have

$$\|A\| = \alpha + \beta - \delta - \gamma > 1.$$

When $\gamma \geq \beta$, by Lemma 6.4, we have

$$\|A\| = \alpha - \beta + \gamma - \delta > 1.$$

If (K2) occurs, then $\gamma' \in [\frac{1}{2}, 1]$ e $-\delta' \in (\frac{1}{2}, 1]$. Therefore,

$$B((1, -1), (0, 1)) = -\delta' + \gamma' > 1$$

and $\|B\| > 1$.

If we have (K3), then

$$A((1, -1), (0, 1)) = -\delta + \gamma > 1,$$

and so $\|A\| > 1$.

If (K4) happens, then $\alpha, \gamma \geq \beta$ and $\alpha, \gamma \geq -\delta$. When $\beta \geq -\delta$, by Lemma 6.3, we have

$$\|A\| = \alpha + \beta + \gamma + \delta > 1.$$

When $-\delta \geq \beta$, by Lemma 6.4, we have

$$\|A\| = \alpha - \beta + \gamma - \delta > 1.$$

The case (4CB) is analogous to (4CA) and (4D) is similar to (4C). □

Theorem 3.2. *The extreme and exposed points of $B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))}$ are the same.*

Proof. It suffices to prove that $\text{ext}B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))} \subseteq \text{exp}B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))}$.

Let us prove that x_1y_1 is an exposed point. Define a linear form $f : \mathcal{L}(^2\ell_\infty^2(\mathbb{R})) \rightarrow \mathbb{R}$ by $f(x_1y_1) = 1$ and $f(x_ry_s) = 0$, for $(r, s) \neq (1, 1)$. Thus $\|f\| = 1 = f(x_1y_1)$ and

$$f(T) < 1$$

for all $T = ax_1y_1 + bx_2y_1 + cx_1y_2 + dx_2y_2$ in $B_{\mathcal{L}(^2\ell_\infty^2(\mathbb{R}))} \setminus \{x_1y_1\}$. In fact, note that $a \leq 1$, otherwise $\|T\| > 1$. If $a = 1$, then $b = 0$, because

$$T((1, 1), (1, 0)) = a + b \text{ and } T((1, -1), (1, 0)) = a - b.$$

The same argument shows that $c = d = 0$. Thus $a < 1$ and $f(T) = a < 1$. A similar argument shows that $\pm x_1 y_1, \pm x_2 y_1, \pm x_1 y_2, \pm x_2 y_2$ are exposed points.

Now, let us prove that $T(x, y) = \frac{1}{2}(x_1 y_1 + x_2 y_1 + x_1 y_2 - x_2 y_2)$ is an exposed point. Define a linear form such that $f(x_1 y_1) = \frac{1}{2}$, $f(x_2 y_1) = \frac{1}{2}$, $f(x_1 y_2) = \frac{1}{2}$ and $f(x_2 y_2) = -\frac{1}{2}$. Thus $f(T) = 1$. Note that $\|f\| = 1$. In fact, by Proposition 2.2, we have

$$\|L\| = \max \{|a + b| + |c + d|, |a - b| + |c - d|\},$$

for all $L(x, y) = ax_1 y_1 + bx_2 y_1 + cx_1 y_2 + dx_2 y_2$. Therefore, if $L \in B_{\mathcal{L}(^2 \ell_\infty^2(\mathbb{R}))}$, then

$$(7) \quad |a + b| \leq 1 \text{ and } |c - d| \leq 1$$

and

$$\begin{aligned} \|f\| &= \sup \{|f(L)| : L \in B_{\mathcal{L}(^2 \ell_\infty^2(\mathbb{R}))}\} \\ &= \sup \left\{ \left| \frac{a + b + c - d}{2} \right| : ax_1 y_1 + bx_2 y_1 + cx_1 y_2 + dx_2 y_2 \in B_{\mathcal{L}(^2 \ell_\infty^2(\mathbb{R}))} \right\} \\ &\leq \frac{1}{2} \sup \{|a + b| + |c - d| : ax_1 y_1 + bx_2 y_1 + cx_1 y_2 + dx_2 y_2 \in B_{\mathcal{L}(^2 \ell_\infty^2(\mathbb{R}))}\} \\ &\leq \frac{1}{2}(1 + 1) = 1. \end{aligned}$$

Moreover, if $f(ax_1 y_1 + bx_2 y_1 + cx_1 y_2 + dx_2 y_2) = 1$ for a certain bilinear form $ax_1 y_1 + bx_2 y_1 + cx_1 y_2 + dx_2 y_2 \in B_{\mathcal{L}(^2 \ell_\infty^2(\mathbb{R}))}$, then

$$\frac{a + b + c - d}{2} = 1$$

and thus

$$(8) \quad \frac{|a + b| + |c - d|}{2} \geq 1.$$

By (7) and (8) we conclude that $|a + b| = 1$ and $|c - d| = 1$. Thus, by Proposition 2.2, we have $|a - b| = 0$ and $|c + d| = 0$ and since $\frac{a+b+c-d}{2} = 1$ we conclude that $a = b = c = -d = \frac{1}{2}$

A similar argument shows that the other extreme points are also exposed points. \square

4. LITTLEWOOD'S 4/3 INEQUALITY AND AN OPTIMIZATION PROBLEM: REAL CASE

Extreme points are important for optimization of convex continuous functions for a very simple reason: first we shall recall a theorem due to Minkowski/Krein-Milman which asserts that if E is a locally convex space and K is a nonempty convex and compact subset of E , then K has at least an extreme point and $K = \overline{\text{conv}}(\text{ext}K)$, where $\text{ext}K$ is the set of all extreme points of K . If $f : K \rightarrow \mathbb{R}$ is a convex continuous function its maximum is attained in an extreme point $k_0 \in K$. In fact, suppose

that $k_0 \in K$ is a point where the maximum is attained; the Minkowski/Krein-Milman asserts that there are $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that

$$k_0 = \sum_{j=1}^n \lambda_j k_j,$$

with $k_1, \dots, k_n \in \text{ext}K$ and $\sum_{j=1}^n \lambda_j = 1$. If the maximum of f is not attained in any extreme point, then

$$f(k_0) \leq \sum_{j=1}^n \lambda_j f(k_j) < \sum_{j=1}^n \lambda_j f(k_0) = f(k_0),$$

a contradiction. However, it is plain that the maximum may also be attained in non-extreme points. For instance, $f : B_{\ell_\infty(\mathbb{R})} \rightarrow \mathbb{R}$ given by $f(x) = \|x\|$ attains its maximum in all points of the unit sphere of $\ell_\infty(\mathbb{R})$ but –for instance– the canonical vectors e_j are not extreme points.

For $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , Littlewood's 4/3 inequality asserts that there exists a sequence of positive scalars $B_2^{\mathbb{K}}$ in $[1, \infty)$ such that

$$\left(\sum_{i,j=1}^{\infty} |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq B_2^{\mathbb{K}} \|U\|$$

for all continuous 2-linear forms $U : c_0 \times c_0 \rightarrow \mathbb{K}$. For $\mathbb{K} = \mathbb{R}$ the optimal constant is $\sqrt{2}$ [10] and for complex scalars all that is known is that

$$1 \leq B_2^{\mathbb{C}} \leq \frac{2}{\sqrt{\pi}}.$$

Littlewood's 4/3 inequality is a forerunner of the classical Bohnenblust–Hille inequality. The Bohnenblust–Hille inequality for m -linear forms ([4]) tells us that there exists a sequence of positive scalars $(B_m^{\mathbb{K}})_{m=1}^{\infty}$ in $[1, \infty)$ such that

$$\left(\sum_{i_1, \dots, i_m=1}^{\infty} |U(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_m^{\mathbb{K}} \|U\|$$

for all continuous m -linear forms $U : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$. The investigation of the optimal constants in the Bohnenblust–Hille inequality can be written as the following optimization problem:

$$B_{\mathbb{K},m} := \inf \left\{ \left(\sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}}, \text{ among all } m\text{-linear forms } T, \text{ with } \|T\| = 1 \right\}.$$

This optimization problem is a rather challenging, still surrounded by many mysteries. For recent developments related to the search of optimal constants we refer to [3, 18] and references therein.

We shall call *optimal bilinear* form any T satisfying the optimization problem above.

By [10] we know that the constant $\sqrt{2}$ is sharp for real scalars when $m = 2$, and it is attained when considering the bilinear form

$$(9) \quad T(x, y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2.$$

The next theorem shows that when considering bilinear forms $T : c_0 \times c_0 \rightarrow \mathbb{R}$ of the form $T(x, y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$ all extremal bilinear forms are very close to (9). It also shows that there are no (norm one) optimal bilinear forms outside the set of extreme points of the closed unit ball of $\mathcal{L}^2(\ell_\infty^2(\mathbb{R}))$.

Theorem 4.1. *Let $T : \ell_\infty^2(\mathbb{R}) \times \ell_\infty^2(\mathbb{R}) \rightarrow \mathbb{R}$ be given by $T(x, y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$, with $a_{ij} \in \mathbb{R}$. Then the bilinear forms satisfying*

$$\left(\sum_{j,k=1}^2 |T(e_j, e_k)|^{\frac{4}{3}} \right)^{\frac{3}{4}} = \sqrt{2} \|T\|$$

are given by

$$T(x, y) = \alpha x_1y_1 + \alpha x_1y_2 + \alpha x_2y_1 - \alpha x_2y_2$$

or

$$T(x, y) = \alpha x_1y_1 + \alpha x_1y_2 - \alpha x_2y_1 + \alpha x_2y_2$$

or

$$T(x, y) = \alpha x_1y_1 - \alpha x_1y_2 + \alpha x_2y_1 + \alpha x_2y_2$$

or

$$T(x, y) = -\alpha x_1y_1 + \alpha x_1y_2 + \alpha x_2y_1 + \alpha x_2y_2$$

for all $\alpha \in \mathbb{R} \setminus \{0\}$.

Proof. If $a_{ij} = 0$ for some $i, j \in \{1, 2\}$ it is not difficult to prove that the constant $\sqrt{2}$ is not achieved.

Let $T : \ell_\infty^2(\mathbb{R}) \times \ell_\infty^2(\mathbb{R}) \rightarrow \mathbb{R}$ be given by $T(x, y) = \sum_{i,j=1}^2 a_{ij}x_iy_j$, with $a_{ij} \in \mathbb{R} \setminus \{0\}$. Let us first suppose that

$$a_{11} \cdot a_{21} > 0 \text{ and } a_{12} \cdot a_{22} > 0.$$

In this case, by Proposition 2.2 we have

$$\|T\| = |a_{11} + a_{21}| + |a_{12} + a_{22}| = \|(a_{11}, a_{21}, a_{12}, a_{22})\|_1.$$

Therefore,

$$\frac{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_{\frac{4}{3}}}{\|T\|} = \frac{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_{\frac{4}{3}}}{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_1} \leq 1.$$

Now, suppose

$$a_{11}, a_{12} > 0 \text{ and } a_{21}, a_{22} < 0$$

Again, using Proposition 2.2, we conclude that

$$\|T\| = |a_{11} - a_{21}| + |a_{12} - a_{22}| = \|(a_{11}, a_{21}, a_{12}, a_{22})\|_1.$$

Thus

$$\frac{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_{\frac{4}{3}}}{\|T\|} = \frac{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_{\frac{4}{3}}}{\|(a_{11}, a_{21}, a_{12}, a_{22})\|_1} \leq 1.$$

The cases

$$a_{11}, a_{12} < 0 \text{ and } a_{21}, a_{22} > 0,$$

$$a_{11}, a_{22} < 0 \text{ and } a_{12}, a_{21} > 0,$$

and

$$a_{11}, a_{22} > 0 \text{ and } a_{12}, a_{21} < 0$$

are similar.

By symmetry, the remaining cases can be summarized in the case

$$a_{11}, a_{12}, a_{21} > 0 \text{ and } a_{22} < 0.$$

By Proposition 2.2 we know that

$$\|T\| = \max\{a_{11} + a_{21} + |a_{12} + a_{22}|, |a_{11} - a_{21}| + a_{12} - a_{22}\}$$

and thus

$$(10) \quad \|T\| = a_{11} + a_{21} + |a_{12} + a_{22}|$$

or

$$(11) \quad \|T\| = |a_{11} - a_{21}| + a_{12} - a_{22}.$$

If (10) occurs, we may have

$$(\alpha) \quad \|T\| = a_{11} + a_{21} + a_{12} + a_{22}$$

or

$$(\beta) \quad \|T\| = a_{11} + a_{21} - a_{12} - a_{22}.$$

If we have (α) , note that

$$a_{12} \geq -a_{22}$$

and define

$$f(x, y, z, w) = \frac{\|(x, y, z, w)\|_{\frac{4}{3}}}{x + y + z + w}.$$

Since

$$f_x(x, y, z, w) = \frac{x^{\frac{1}{3}}(x + y + z + w) - \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(x + y + z + w)^2},$$

$$f_y(x, y, z, w) = \frac{y^{\frac{1}{3}}(x + y + z + w) - \|(x, y, z, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{\|(x, y, z, w)\|^{\frac{1}{3}}_{\frac{4}{3}}(x + y + z + w)^2},$$

$$f_z(x, y, z, w) = \frac{z^{\frac{1}{3}}(x + y + z + w) - \|(x, y, z, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{\|(x, y, z, w)\|^{\frac{1}{3}}_{\frac{4}{3}}(x + y + z + w)^2},$$

$$f_w(x, y, z, w) = \frac{w^{\frac{1}{3}}(x + y + z + w) - \|(x, y, z, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{\|(x, y, z, w)\|^{\frac{1}{3}}_{\frac{4}{3}}(x + y + z + w)^2},$$

we have

$$\nabla f = 0 \Rightarrow x^{\frac{1}{3}} = y^{\frac{1}{3}} = z^{\frac{1}{3}} = w^{\frac{1}{3}} \Rightarrow x = y = z = w.$$

Since $x, y, z > 0$ and $w < 0$, we conclude that $\nabla f \neq 0$.

From now on let $(a, b, c, d) = (a_{11}, a_{21}, a_{12}, a_{22})$. By Lemma 6.3

$$\begin{aligned} & \{(a, b, c, d) : a, b, c, -d > 0 : a + b + c + d \geq |a - b| + c - d\} = \\ & \{(a, b, c, d) : a, b, c, -d > 0 : c \geq -d, b \geq -d, a \geq -d\} \\ & \cup \{(a, b, c, d) : a, b, c, -d > 0 : a \geq c, b \geq c, -d \geq c\}. \end{aligned}$$

Since in our case $c \geq -d$ we conclude that we shall search the maximum of f in the set

$$H := \{(a, b, c, d) : a, b, c > 0, d < 0 \text{ and } c \geq -d, b \geq -d, a \geq -d\}.$$

Defining

$$A := \{(a, b, c, d) : a, b, c > 0, d < 0 \text{ and } a > -d, b > -d, c > -d\},$$

the maximum of f belongs to $H \setminus A$. Note that

$$\begin{aligned} H \setminus A &= \{(a, b, c, -a) : a, b, c > 0 \text{ and } b \geq a, c \geq a\} \\ &\cup \{(a, b, c, -b) : a, b, c > 0 \text{ and } a \geq b, c \geq b\} \\ &\cup \{(a, b, c, -c) : a, b, c > 0 \text{ and } a \geq c, b \geq c\}. \end{aligned}$$

Let us first consider the set

$$\{(a, b, c, -a) : a, b, c > 0 \text{ and } b \geq a, c \geq a\}.$$

In this case, let

$$g(x, y, z) := f(x, y, z, -x) = \frac{(2x^{\frac{4}{3}} + y^{\frac{4}{3}} + z^{\frac{4}{3}})^{\frac{3}{4}}}{y + z}.$$

Thus

$$g_x = \frac{2x^{\frac{1}{3}}}{\|(x, y, z, -x)\|_{\frac{4}{3}}^{\frac{1}{3}}(y+z)}.$$

and

$$g_x = 0 \Leftrightarrow x = 0.$$

Hence, the maximum of g does not belong to $\{(a, b, c, -a) : a, b, c > 0 \text{ and } b > a, c > a\}$, i.e., the maximum belongs to

$$\begin{aligned} & \{(a, b, c, -a) : a, b, c > 0 \text{ and } b \geq a, c \geq a\} \setminus \{(a, b, c, -a) : a, b, c > 0 \text{ and } b > a, c > a\} = \\ & \{(a, a, c, -a) : a, c > 0 \text{ and } c \geq a\} \cup \{(a, b, a, -a) : a, b > 0 \text{ and } b \geq a\}. \end{aligned}$$

Considering the set $\{(a, a, c, -a) : a, c > 0 \text{ and } c \geq a\}$, we define

$$h(x, z) := f(x, x, z, -x) = \frac{(3x^{\frac{4}{3}} + z^{\frac{4}{3}})^{\frac{3}{4}}}{x+z}.$$

and

$$\begin{aligned} h_x &= \frac{3x^{\frac{1}{3}} - \|(x, x, z, -x)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, x, z, -x)\|_{\frac{4}{3}}^{\frac{1}{3}}(x+z)^2}, \\ h_z &= \frac{z^{\frac{1}{3}} - \|(x, x, z, -x)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, x, z, -x)\|_{\frac{4}{3}}^{\frac{1}{3}}(x+z)^2}. \end{aligned}$$

Thus

$$\nabla h = 0 \Leftrightarrow (x, z) = \left(\frac{1}{28}, \frac{27}{28}\right)$$

and hence $h\left(\frac{1}{28}, \frac{27}{28}\right) < \sqrt{2}$. Note that

$$f(x, x, x, -x) = \frac{\|(x, x, x, -x)\|_{\frac{4}{3}}^{\frac{4}{3}}}{x+x+x-x} = \frac{2^{\frac{3}{2}}x}{2x} = \sqrt{2}.$$

The other cases are similar.

Now we consider the case (β) . Defining

$$f(x, y, z, w) := \frac{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{x+y-z-w},$$

we have

$$\begin{aligned} f_x(x, y, z, w) &= \frac{x^{\frac{1}{3}}(x+y-z-w) - \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(x+y-z-w)^2}, \\ f_y(x, y, z, w) &= \frac{y^{\frac{1}{3}}(x+y-z-w) - \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(x+y-z-w)^2}, \end{aligned}$$

$$f_z(x, y, z, w) = \frac{z^{\frac{1}{3}}(x + y - z - w) + \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(x + y - z - w)^2},$$

and

$$f_w(x, y, z, w) = \frac{w^{\frac{1}{3}}(x + y - z - w) + \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}}}{\|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(x + y - z - w)^2}.$$

Since $z, (x + y - z - w), \|(x, y, z, w)\|_{\frac{4}{3}} > 0$, we have

$$z^{\frac{1}{3}}(x + y - z - w) + \|(x, y, z, w)\|_{\frac{4}{3}}^{\frac{4}{3}} > 0,$$

and $\nabla f \neq 0$. Note that in (β) we have $c \leq -d$ and, by Lemma 6.3, we have

$$\begin{aligned} H_1 &:= \{(a, b, c, d) : a, b, c > 0, d < 0, a + b - c - d \geq |a - b| + c - d\} \\ &= \{(a, b, c, d) : a, b, c > 0, d < 0, -d \geq c, b \geq c, a \geq c\}. \end{aligned}$$

So, if $(x, y, z, w) \in A_1 := \{(a, b, c, d) : a, b, c > 0, d < 0, -d > c, b > c, a > c\}$, then $\nabla f(x, y, z, w) \neq 0$, and the maximum of f does not belong to A . We conclude that the maximum of f belongs to

$$\begin{aligned} H_1 \setminus A &= \{(a, b, a, d) : a, b > 0, d < 0 \text{ and } b \geq a, -d \geq a\} \\ &\cup \{(a, b, b, d) : a, b > 0, d < 0 \text{ and } a \geq b, -d \geq b\} \\ &\cup \{(a, b, c, -c) : a, b, c > 0 \text{ and } a \geq c, b \geq c\}. \end{aligned}$$

For $\{(a, b, a, d) : a, b > 0, d < 0 \text{ and } b \geq a, -d \geq a\}$, we define

$$g(x, y, w) := \frac{(2x^{\frac{4}{3}} + y^{\frac{4}{3}} + w^{\frac{4}{3}})^{\frac{3}{4}}}{y - w}$$

and

$$g_x = \frac{2x^{\frac{1}{3}}}{\|(x, y, x, w)\|_{\frac{4}{3}}^{\frac{1}{3}}(y - w)}.$$

We thus conclude that

$$g_x = 0 \Leftrightarrow x = 0$$

and $\nabla g \neq 0$ for all points of $\{(a, b, a, d) : a, b > 0, d < 0 \text{ and } b > a, -d > a\}$. Hence, the maximum of g belongs to

$$\begin{aligned} &\{(a, b, a, d) : a, b > 0, d < 0 \text{ and } b \geq a, -d \geq a\} \setminus \{(a, b, a, d) : a, b > 0, d < 0 \text{ and } b > a, -d > a\} = \\ &\{(a, a, a, d) : a > 0, d < 0 \text{ and } -d \geq a\} \cup \{(a, b, a, -a) : a, b > 0 \text{ and } b \geq a\}. \end{aligned}$$

For the set $\{(a, a, a, d) : a > 0, d < 0 \text{ and } -d \geq a\}$, we define

$$h(x, w) := \frac{(3x^{\frac{4}{3}} + w^{\frac{4}{3}})^{\frac{3}{4}}}{x - w}.$$

Note that

$$h_x = \frac{3x^{\frac{1}{3}} - \|(x, x, x, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{\|(x, x, x, w)\|^{\frac{1}{3}}_{\frac{4}{3}}(x - w)^2}$$

and

$$h_w = \frac{w^{\frac{1}{3}} + \|(x, x, x, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{\|(x, x, x, w)\|^{\frac{1}{3}}_{\frac{4}{3}}(x - w)^2}.$$

Thus

$$3x^{\frac{1}{3}} - \|(x, x, x, w)\|^{\frac{4}{3}}_{\frac{4}{3}} = 0 \text{ and } w^{\frac{1}{3}} + \|(x, x, x, w)\|^{\frac{4}{3}}_{\frac{4}{3}} = 0 \Leftrightarrow (x, w) = (0, 0).$$

Hence the maximum of f is obtained in

$$f(x, x, x, -x) = \sqrt{2}.$$

The other cases are similar.

The case $\|T\| = |a_{11} - a_{21}| + a_{12} - a_{22}$ is divided in two cases

$$(\gamma) \quad \|T\| = a_{11} - a_{21} + a_{12} - a_{22},$$

$$(\delta) \quad \|T\| = -a_{11} + a_{21} + a_{12} - a_{22}.$$

If (γ) holds, we consider

$$f(x, y, z, w) = \frac{\|(x, y, z, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{x - y + z - w},$$

and if (δ) holds we consider

$$f(x, y, z, w) = \frac{\|(x, y, z, w)\|^{\frac{4}{3}}_{\frac{4}{3}}}{-x + y + z - w}.$$

For these cases, using Lemma 6.4 we prove that the maximum is attained when $x = y = z = -w$. \square

5. THE COMPLEX CASE: NUMERICAL AND ANALYTICAL CONSIDERANTIONS

It is well known (see [16]) that for complex scalars we have

$$\left(\sum_{i,j=1}^{\infty} |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \frac{2}{\sqrt{\pi}} \|U\|$$

for all continuous m -linear forms $U : c_0 \times c_0 \rightarrow \mathbb{C}$, but it is unknown if the constant $\frac{2}{\sqrt{\pi}}$ is sharp.

Since the optimal constant of the Littlewood's $4/3$ inequality is achieved when considering simple looking bilinear forms with only four monomials, it is natural to begin the investigation of the complex

case in a similar setting. We have strong numerical evidence that considering $T : c_0 \times c_0 \rightarrow \mathbb{C}$ given by $T(z, w) = \sum_{i,j=1}^2 a_{ij} z_i w_j$ with $a_{ij} \in \mathbb{R}$ the optimal constant is 1. For instance, we can consider a discretized region S in $[-1, 1]^4$ such that for any $T(z, w) = \sum_{i,j=1}^2 a_{ij} z_i w_j$ with $a_{ij} \in [-1, 1]$, there are $s_1, s_2, s_3, s_4 \in S$ such that

$$|a_{11} - s_1|, |a_{12} - s_2|, |a_{21} - s_3|, |a_{22} - s_4| < 10^{-1},$$

and, for all such s_1, s_2, s_3, s_4 we have

$$\left(\sum_{i,j=1}^2 |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \|U\|,$$

for $U(z, w) = s_1 z_1 w_1 + s_2 z_1 w_2 + s_3 z_2 w_1 + s_4 z_2 w_2$.

The following theorem gives a formal proof that in several cases the optimal constant is in fact 1 :

Theorem 5.1. *Let $T : c_0 \times c_0 \rightarrow \mathbb{C}$ be given by $T(z, w) = \sum_{i,j=1}^2 a_{ij} z_i w_j$ with $a_{ij} \in \mathbb{R}$. Then*

$$\left(\sum_{i,j=1}^2 |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq \|T\|,$$

when

- (1) $a_{11}a_{21} = 0$ or $a_{12}a_{22} = 0$;
- (2) $a_{11}a_{21} > 0$ and $a_{12}a_{22} > 0$;
- (3) $a_{11}a_{21} < 0$ and $a_{12}a_{22} < 0$;
- (4) $a_{11}a_{21} > 0$ and $a_{12}a_{22} < 0$ and $a_{11}a_{21} + a_{12}a_{22} = 0$;
- (5) $a_{11}a_{21} < 0$ and $a_{12}a_{22} > 0$ and $a_{11}a_{21} + a_{12}a_{22} = 0$.

Proof. For the case (1), there is no loss of generality in supposing $a_{11} = 0$. By Proposition 2.1, we have

$$\|T\| = \max\{|a_{21}| + |a_{12} + a_{22}|, |a_{21}| + |a_{12} - a_{22}|\}.$$

If $a_{12}a_{22} \geq 0$, then $\|T\| = |a_{21}| + |a_{12} + a_{22}| = \|(a_{11}, a_{12}, a_{21}, a_{22})\|_1$. Now, if $a_{12}a_{22} \leq 0$, then $\|T\| = |a_{21}| + |a_{12} - a_{22}| = \|(a_{11}, a_{12}, a_{21}, a_{22})\|_1$. Therefore

$$\frac{\left(\sum_{i,j=1}^2 |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}}}{\|T\|} \leq 1.$$

For the case (2), by Proposition 2.1, we have

$$\begin{aligned} \|T\| &= \max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\} \\ &= |a_{11} + a_{21}| + |a_{12} + a_{22}| = \|(a_{11}, a_{12}, a_{21}, a_{22})\|_1, \end{aligned}$$

and, again,

$$\frac{\left(\sum_{i,j=1}^2 |T(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{\|T\|} \leq 1.$$

Considering (3), again Proposition 2.1 tells us that

$$\begin{aligned} \|T\| &= \max\{|a_{11} + a_{21}| + |a_{12} + a_{22}|, |a_{11} - a_{21}| + |a_{12} - a_{22}|\} \\ &= |a_{11} - a_{21}| + |a_{12} - a_{22}| = \|(a_{11}, a_{12}, a_{21}, a_{22})\|_1, \end{aligned}$$

and

$$\frac{\left(\sum_{i,j=1}^2 |T(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{\|T\|} \leq 1.$$

Now we deal with the case (4). There is no loss of generality in supposing $a_{11}, a_{21}, a_{21} > 0$ and $a_{22} < 0$. Since $a_{12}a_{22} = -a_{11}a_{21}$ we have

$$2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right) = 4a_{11}a_{21} = -4a_{12}a_{22}$$

and

$$\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2) = a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2),$$

and we obtain

$$\frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right)} = \frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}}.$$

If

$$\left| \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right)} \right| \leq 1,$$

then

$$a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}} \geq 0$$

and

$$\begin{aligned} (12) \quad \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21} \frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}}} &= \sqrt{a_{11}^2 + a_{21}^2 + \frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{2}} \\ &= \sqrt{\frac{a_{12}^2 + a_{22}^2 + a_{11}^2 + a_{21}^2}{2}} \\ &= 2^{-\frac{1}{2}} \sqrt{a_{12}^2 + a_{22}^2 + a_{11}^2 + a_{21}^2}. \end{aligned}$$

A similar reasoning tells us that

$$(13) \quad \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22} \frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}}} = 2^{-\frac{1}{2}} \sqrt{a_{12}^2 + a_{22}^2 + a_{11}^2 + a_{21}^2}.$$

Summing up (12) and (13) we obtain

$$\begin{aligned} \sqrt{a_{11}^2 + a_{21}^2 + 2a_{11}a_{21}\frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}}} + \sqrt{a_{12}^2 + a_{22}^2 + 2a_{12}a_{22}\frac{a_{12}^2 + a_{22}^2 - (a_{11}^2 + a_{21}^2)}{4a_{11}a_{21}}} \\ = 2^{\frac{1}{2}}\sqrt{a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2}. \end{aligned}$$

By Proposition 2.1 we conclude that

$$\|T\| = 2^{\frac{1}{2}}\sqrt{a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2}.$$

Therefore, by the Hölder inequality we have

$$\|(a_{11}, a_{12}, a_{21}, a_{22})\|_{\frac{4}{3}} \leq 2^{\frac{1}{2}}\|(a_{11}, a_{12}, a_{21}, a_{22})\|_2 = \|T\|.$$

Recall that by Lemma 6.5, we have

$$\left| \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right)} \right| \geq 1$$

if, and only if,

$$\left| \frac{a_{12}a_{22}}{a_{11}a_{21}}(a_{11} - a_{21}) \right| \geq a_{12} - a_{22} \text{ or } \left| \frac{a_{11}a_{21}}{a_{12}a_{22}}(a_{12} + a_{22}) \right| \geq a_{11} + a_{21}.$$

Since $a_{11}a_{21} + a_{12}a_{22} = 0$, it follows that

$$(a_{21}, a_{22}) \in \mathbb{R}(a_{12}, -a_{11}),$$

i.e., $a_{21} = ta_{12}$ and $a_{22} = -ta_{11}$ with $t > 0$. Thus, by Proposition 2.1 we have

$$\|T\| = \max\{a_{11} + ta_{12} + |a_{12} - ta_{11}|, |a_{11} - ta_{12}| + a_{12} + ta_{11}\}.$$

Note that we have two cases:

$$(a.1) \quad \|T\| = a_{11} + ta_{12} + |a_{12} - ta_{11}|;$$

$$(a.2) \quad \|T\| = |a_{11} - ta_{12}| + a_{12} + ta_{11}.$$

In the case (a.1), by Lemma 6.3, we have

$$(14) \quad a_{11} \geq -a_{22}, \quad a_{21} \geq -a_{22}, \quad a_{12} \geq -a_{22}$$

or

$$(15) \quad a_{11} \geq a_{12}, \quad a_{21} \geq a_{12}, \quad -a_{22} \geq a_{12}.$$

Let us suppose (14). Note that

$$a_{11} \geq -a_{22}, \quad a_{21} \geq -a_{22}, \quad a_{12} \geq -a_{22} \Leftrightarrow 1 \geq t, \quad a_{12} \geq a_{11}, \quad a_{12} \geq ta_{11}.$$

In this case,

$$\|T\| = (1-t)a_{11} + (1+t)a_{12}.$$

Since $0 < t \leq 1$, we have

$$t-1 \leq 0 \Rightarrow (t-1)a_{12} \leq (t+1)a_{11} \Rightarrow -a_{12} - ta_{11} \leq a_{11} - ta_{12}$$

and

$$1-t \leq 1+t \Rightarrow (1-t)a_{11} \leq (1+t)a_{12} \Rightarrow a_{11} - ta_{12} \leq a_{12} + ta_{11}.$$

We thus conclude that

$$(16) \quad |a_{11} - ta_{12}| \leq a_{12} + ta_{11}.$$

Note that

$$(17) \quad (t-1)a_{11} \leq (1+t)a_{12} \Rightarrow -a_{11} - ta_{12} \leq a_{12} - ta_{11}.$$

and also that

$$(18) \quad a_{12} - ta_{11} \leq a_{11} + ta_{12} \Leftrightarrow a_{12} - a_{11} \leq t(a_{11} + a_{12}) \Leftrightarrow \frac{a_{12} - a_{11}}{a_{11} + a_{12}} \leq t.$$

If $a_{11} = a_{12}$, then combining the information of (17) and (18) we have

$$|a_{12} - ta_{11}| \leq a_{11} + ta_{12}$$

and by Lemma 6.5 we conclude that

$$(19) \quad \left| \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11} a_{21} \left(1 - \frac{a_{11} a_{21}}{a_{12} a_{22}}\right)} \right| \leq 1.$$

By (19), using the previous estimates the proof is completed for this case.

Now suppose

$$a_{11} < a_{12} \text{ and } t \leq \frac{a_{12} - a_{11}}{a_{11} + a_{12}}.$$

By (18), we have

$$a_{11} + ta_{12} \leq a_{12} - ta_{11}.$$

In this case, fixing $a_{11}, a_{12} > 0$, we define $f : [0, \frac{a_{12}-a_{11}}{a_{11}+a_{12}}] \rightarrow \mathbb{R}$ by

$$f(t) = \frac{\left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) \right)^{\frac{3}{4}}}{\|T\|}.$$

Since

$$f(0) = \frac{\left(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}} \right)^{\frac{3}{4}}}{a_{11} + a_{12}} \leq 1$$

and

$$f\left(\frac{a_{12} - a_{11}}{a_{11} + a_{12}}\right) \leq 1,$$

we have

$$f'(t) = \frac{t^{\frac{1}{3}}(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})((1-t)a_{11} + (1+t)a_{12}) - \left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})\right)(a_{12} - a_{11})}{\left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})\right)^{\frac{1}{4}}((1-t)a_{11} + (1+t)a_{12})^2}.$$

Thus, $f'(t_0) = 0$ if, and only if,

$$t_0 = -\frac{(a_{11} - a_{12})^3}{(a_{11} + a_{12})^3}.$$

But t_0 is a point of minimum for f and thus $\max f \leq 1$.

Now we consider the case (15).

Note that

$$a_{11} \geq a_{12}, a_{21} \geq a_{12}, -a_{22} \geq a_{12} \Leftrightarrow a_{11} \geq a_{12}, t \geq 1, ta_{11} \geq a_{12}$$

and in this case

$$\|T\| = (1+t)a_{11} + (t-1)a_{12}.$$

Since $1 \leq t$, we have

$$(20) \quad (t-1) \leq (t+1) \Rightarrow (t-1)a_{11} \leq (t+1)a_{12} \Rightarrow -a_{11} - ta_{12} \leq a_{12} - ta_{11}.$$

and

$$(21) \quad 1-t \leq 0 \Rightarrow (1-t)a_{12} \leq (1+t)a_{11} \Rightarrow a_{12} - ta_{11} \leq a_{11} + ta_{12}.$$

Thus, by (20) and (21), we have

$$|a_{12} - ta_{11}| \leq a_{11} + ta_{12}.$$

Note also that

$$(22) \quad (1-t)a_{11} \leq (1+t)a_{12} \Rightarrow a_{11} - ta_{12} \leq a_{12} + ta_{11}$$

and

$$(23) \quad -a_{12} - ta_{11} \leq a_{11} - ta_{12} \Leftrightarrow -a_{11} - a_{12} \leq t(a_{12} - a_{11}).$$

If $a_{12} = a_{11}$, by (20) and (21), we have

$$|a_{11} - ta_{12}| \leq a_{12} + ta_{11}$$

and thus

$$\left| \frac{\frac{a_{11}^2 a_{21}^2}{a_{12}^2 a_{22}^2} (a_{12}^2 + a_{22}^2) - (a_{11}^2 + a_{21}^2)}{2a_{11}a_{21} \left(1 - \frac{a_{11}a_{21}}{a_{12}a_{22}}\right)} \right| \leq 1$$

and the proof of this case is done. If $a_{12} < a_{11}$ and $\frac{a_{11}+a_{12}}{a_{11}-a_{12}} \leq t$ then, by (23) we have

$$a_{11} - ta_{12} \leq -a_{12} - ta_{11}.$$

Fixing $a_{11}, a_{12} > 0$, we consider $g : \left[\frac{a_{11}+a_{12}}{a_{11}-a_{12}}, \infty \right) \rightarrow \mathbb{R}$ by

$$g(t) = \frac{\left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) \right)^{\frac{3}{4}}}{\|T\|}.$$

Note that

$$g\left(\frac{a_{11}+a_{12}}{a_{11}-a_{12}}\right) \leq 1$$

and

$$g(t) = \frac{t \left((a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) + \frac{(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})}{t^{\frac{4}{3}}} \right)^{\frac{3}{4}}}{t((a_{11} + a_{12}) + \frac{a_{11}-a_{12}}{t})} = \frac{\left((a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) + \frac{(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})}{t^{\frac{4}{3}}} \right)^{\frac{3}{4}}}{((a_{11} + a_{12}) + \frac{a_{11}-a_{12}}{t})},$$

and

$$\lim_{t \rightarrow \infty} g(t) = \frac{\left((a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) \right)^{\frac{3}{4}}}{a_{11} + a_{12}} \leq 1.$$

Moreover,

$$g'(t) = \frac{t^{\frac{1}{3}}(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}})((1+t)a_{11} + (t-1)a_{12}) - \left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) \right) (a_{11} + a_{12})}{\left((1+t^{\frac{4}{3}})(a_{11}^{\frac{4}{3}} + a_{12}^{\frac{4}{3}}) \right)^{\frac{1}{4}} ((1+t)a_{11} + (t-1)a_{12})^2}.$$

Therefore $g'(t_0) = 0$ if, and only if,

$$t_0 = \frac{(a_{11} + a_{12})^3}{(a_{11} - a_{12})^3},$$

and t_0 is a point of minimum for g and we conclude that $\max g \leq 1$.

The case (a.2) is similar and (5) is analogous to (4). □

Corollary 5.2. *Let $\alpha \neq 0$. Then $T : \ell_\infty^2(\mathbb{C}) \times \ell_\infty^2(\mathbb{C}) \rightarrow \mathbb{C}$ given by*

$$T(x, y) = \alpha z_1 w_1 + \alpha z_1 w_2 + \alpha z_2 w_1 - \alpha z_2 w_2$$

or

$$T(x, y) = \alpha z_1 w_1 + \alpha z_1 w_2 - \alpha z_2 w_1 + \alpha z_2 w_2$$

or

$$T(x, y) = \alpha z_1 w_1 - \alpha z_1 w_2 + \alpha z_2 w_1 + \alpha z_2 w_2$$

or

$$T(x, y) = -\alpha z_1 w_1 + \alpha z_1 w_2 + \alpha z_2 w_1 + \alpha z_2 w_2$$

satisfies

$$\left(\sum_{i,j=1}^2 |T(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} = \|T\|.$$

6. APPENDIX: SOME ELEMENTARY LEMMATA

This section is entirely devoted to five elementary lemmata used in the paper. The first two lemmata are quite simple and we omit their proof.

Lemma 6.1. *Let $a, b, c > 0$ and $d < 0$ be such that*

$$|a + b| + |c + d| = |a - b| + |c - d|.$$

If $a \neq b$, then $b = -d$ or $a = -d$.

Lemma 6.2. *Let $a, b \in \mathbb{R} \setminus \{0\}$.*

(a) If $|t_0| \leq 1$, then $|a + bt_0| \leq |a + b|$ or $|a + bt_0| \leq |a - b|$.

(b) The maximum of the function $f : [-1, 1] \rightarrow \mathbb{R}$ given by

$$f(t) = |a + bt| + |c + dt|$$

occurs for $t = -1$ or $t = 1$.

Lemma 6.3. *Let $a, b, c > 0$ and $d < 0$. Then*

$$a + b + |c + d| \geq |a - b| + c - d$$

if, and only if,

$$a \geq -d, \quad b \geq -d, \quad c \geq -d \quad \text{or} \quad a \geq c, \quad b \geq c, \quad -d \geq c.$$

Proof. Suppose that $a, b, c > 0$ and $d < 0$ are such that

$$a + b + |c + d| \geq |a - b| + c - d.$$

If $c \geq -d$, we have

$$a + b + c + d \geq |a - b| + c - d.$$

Since $a - b \leq |a - b|$ and $b - a \leq |b - a| = |a - b|$, we have

$$a + b + c + d \geq a - b + c - d$$

and thus $b \geq -d$. We also have

$$a + b + c + d \geq b - a + c - d$$

and thus $a \geq -d$.

Now suppose that $-d \geq c$. Then

$$a + b - d - c \geq a - b + c - d$$

and hence $b \geq c$. Besides,

$$a + b - d - c \geq b - a + c - d$$

and thus $a \geq c$.

Reciprocally, suppose that $a, b, c > 0$ and $d < 0$ with $a \geq -d$, $b \geq -d$, $c \geq -d$. Then

$$b \geq -d \Rightarrow 2b \geq -2d \Rightarrow b + d \geq -b - d \Rightarrow a + b + c + d \geq a - b + c - d$$

and

$$a \geq -d \Rightarrow 2a \geq -2d \Rightarrow a + d \geq -a - d \Rightarrow a + b + c + d \geq b - a + c - d.$$

Hence

$$a + b + c + d \geq |a - b| + c - d.$$

Now, suppose that $a, b, c > 0$ and $d < 0$, with $a \geq c$, $b \geq c$, $-d \geq c$. Then

$$b \geq c \Rightarrow 2b \geq 2c \Rightarrow b - c \geq -b + c \Rightarrow a + b - c - d \geq a - b + c - d$$

and

$$a \geq c \Rightarrow 2a \geq 2c \Rightarrow a - c \geq -a + c \Rightarrow a + b - c - d \geq b - a + c - d.$$

and we finally obtain

$$a + b - c - d \geq |a - b| + c - d.$$

□

The next lemma has a proof similar to the proof of the previous lemma, and we omit it.

Lemma 6.4. *Let $a, b, c > 0$ and $d < 0$. Then*

$$|a - b| + c - d \geq a + b + |c + d|$$

if, and only if,

$$a \geq b, c \geq b, -d \geq b \text{ or } b \geq a, c \geq a, -d \geq a.$$

Lemma 6.5. *Let $a, b, c > 0$ and $d < 0$. Then*

$$\left| \left(\frac{ab}{cd} \right)^2 (c^2 + d^2) - (a^2 + b^2) \right| \leq 2ab \left(1 - \frac{ad}{cd} \right)$$

if, and only if,

$$\left| \frac{cd}{ab} (a - b) \right| \leq c - d \text{ and } \left| \frac{ab}{cd} (c + d) \right| \leq a + b.$$

Proof. Note that

$$\left| \left(\frac{ab}{cd} \right)^2 (c^2 + d^2) - (a^2 + b^2) \right| \leq 2ab \left(1 - \frac{ad}{cd} \right)$$

if, and only if,

$$-2ab \left(1 - \frac{ad}{cd} \right) \leq \left(\frac{ab}{cd} \right)^2 (c^2 + d^2) - (a^2 + b^2) \leq 2ab \left(1 - \frac{ad}{cd} \right).$$

Moreover

$$-2ab \left(1 - \frac{ad}{cd} \right) \leq \left(\frac{ab}{cd} \right)^2 (c^2 + d^2) - (a^2 + b^2)$$

if, and only if

$$\left| \frac{cd}{ab} (a - b) \right| \leq c - d.$$

In a similar fashion we show that the other inequality is equivalent to

$$\left| \frac{ab}{cd} (c + d) \right| \leq a + b.$$

□

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